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A remark on $\mathscr{C}^\infty$ definable equivalence

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Abstract. We establish that if a submanifold M of $\mathbb{R}^n$ is definable in some o-minimal structure then any definable submanifold $N \subset \mathbb{R}^n$ which is $\mathscr{C}^\infty$ diffeomorphic to M, with a diffeomorphism $h : N \to M$ that is sufficiently close to the identity, must be $\mathscr{C}^\infty$ definably diffeomorphic to M. The definable diffeomorphism between N and M is then provided by a tubular neighborhood of M.

1. Introduction. The framework of o-minimal structures is well adapted to both analysis and geometry, and definable mappings have many finiteness properties that are valuable for applications. It is however not always easy to construct definable diffeomorphisms. For instance, smooth trivializations are often generated by integration of a vector field [2] and the flow of a definable vector field may fail to be definable in the same structure. Furthermore, M. Shiota gave examples of algebraic smooth manifolds that are $\mathscr{C}^\infty$ diffeomorphic but not Nash diffeomorphic (i.e. not semialgebraically $\mathscr{C}^\infty$ equivalent) [4]. This points out that $\mathscr{C}^\infty$ equivalence of definable manifolds does not guarantee definable $\mathscr{C}^\infty$ equivalence.

In this note, we show that such pathologies cannot arise if the diffeomorphisms considered are sufficiently close to the identity. Namely, we establish that if a submanifold M of $\mathbb{R}^n$ is definable in some o-minimal structure then any definable submanifold $N \subset \mathbb{R}^n$ which is $\mathscr{C}^\infty$ diffeomorphic to M, with a diffeomorphism $h : N \to M$ that is sufficiently close to the identity, must be definably $\mathscr{C}^\infty$ diffeomorphic to M (Theorem 2.1). The definable diffeomorphism between N and M is then provided by a tubular neighborhood of M.

We briefly recall that an *o-minimal structure* expanding the real field $(\mathbb{R}, +, \cdot)$ is the data for every n of a Boolean algebra $\mathcal{D}_n$ of subsets of $\mathbb{R}^n$ containing all the algebraic subsets of $\mathbb{R}^n$ and satisfying the following axioms:

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(1) If $A \in \mathcal{D}_m$, $B \in \mathcal{D}_n$, then $A \times B \in \mathcal{D}_{m+n}$.
(2) If $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the natural projection and $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$.
(3) $\mathcal{D}_1$ is nothing else but all the finite unions of points and intervals.

A set belonging to the structure $\mathcal{D}$ is called a *definable set* and a map whose graph is in the structure $\mathcal{D}$ is called a *definable map*. Given $B \subset \mathbb{R}^k$, we say that $(Z_t)_{t\in B}$ is a *definable family* of subsets of $\mathbb{R}^n$ if for each $t \in B$, $Z_t \subset \mathbb{R}^n$ and $\bigcup_{t\in B}\{t\} \times Z_t \in \mathcal{D}_{k+n}$ (in particular, Z_t is definable for all t). A family $(\varphi_t)_{t\in B}$ of mappings is said to be definable if the family of their graphs is a definable family of sets.

Given a definable set $A \subset \mathbb{R}^n$, we denote by $\mathcal{D}^+(A)$ the set of positive definable continuous functions on A.

Given a mapping $f : A \to B$ with $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^k$, we denote by $|f|$ the function which assigns to $x \in A$ the number $|f(x)|$. Here we stress that since $|f|$ is not a real number but a function, this does not define a norm on the space of mappings.

If f is a differentiable mapping, we denote by $d_x f$ its derivative at x and we will write $|d_x f|$ for the norm of $d_x f$ (as a linear mapping) derived from the euclidean norm $|\cdot|$. We will write $|df|$ for the function defined by $x \mapsto |d_x f|$. We then set

$$|f|_1 := |f| + |df|.$$

We will denote by $B(x,r)$ the open euclidean ball of radius r centered at x, and by $\overline{A}$ the closure of A in the euclidean topology.

2. Definable diffeomorphisms via retractions. Let us recall that given a definable $\mathscr{C}^p$ submanifold M of $\mathbb{R}^n$, $p \geq 2$, there is a definable neighborhood U of M in $\mathbb{R}^n$ and a definable retraction $r : U \to M$ such that for all $x \in U$, $r(x)$ is the point that realizes the distance from x to M. The vector $x - r(x)$ is then orthogonal to the tangent space to M at $r(x)$ and we say that (U, r) is a *tubular neighborhood* of M. The mapping r is at least $\mathscr{C}^{p-1}$ and, if M is $\mathscr{C}^\infty$, then so is r [3], [5, Proposition 2.4.1]; see also [1, Theorem 6.11].

THEOREM 2.1. *Let $M \subset \mathbb{R}^n$ be a closed definable $\mathscr{C}^2$ submanifold and let (U, r) be a definable tubular neighborhood of M. There exists $\varepsilon \in \mathcal{D}^+(\mathbb{R}^n)$ such that if $N \subset \mathbb{R}^n$ is any definable $\mathscr{C}^2$ submanifold for which there exists a $\mathscr{C}^1$ diffeomorphism $h : N \to M$ satisfying $|h - \mathrm{id}_{|N}|_1 < \varepsilon$ then N is contained in U and the restriction $r_{|N} : N \to M$ is a $\mathscr{C}^1$ diffeomorphism.*

Proof. Let $N \subset \mathbb{R}^n$ be a definable $\mathscr{C}^2$ submanifold and $h : N \to M$ a diffeomorphism such that $|h(x) - x|_1 < \varepsilon(x)$ for some $\varepsilon \in \mathcal{D}^+(\mathbb{R}^n)$. We assume $\varepsilon < 1$ and we will put extra requirements on ε on the way. For

$\delta \in \mathcal{D}^+(M)$ define

$$U_\delta := \{x \in U : \mathrm{dist}(x, M) < \delta(r(x))\}, \tag{2.1}$$

where $\mathrm{dist}(x, M)$ is the euclidean distance of x to M. If δ is sufficiently small we have $\overline{U_\delta} \subset U$. Replacing U with U_δ for some small δ, we can assume that $|d_x r|$ is bounded on bounded sets. Moreover, the assumption $|h - \mathrm{id}_{|N}|_1 < \varepsilon$ entails $\mathrm{dist}(x, M) < \varepsilon(x)$ for every x in N and therefore $N \subset U_\delta$ if $\varepsilon(x) \le \eta(x) := \inf\{\delta(z)/2 : |z| < |x| + 1\}$.

Let $x \in N$ and take a unit vector $u \in T_x N$. Put $v := d_x h(u)$. By assumption, we have $|u - v| < \varepsilon(x)$. Observe that

(1) $|d_x r(u) - d_x r(v)| \le |u - v|\, |d_x r|$,
(2) $|d_{h(x)} r(v) - u| = |v - u|$ (as r is the identity on M),
(3) by continuity of $x \mapsto d_x r$, there is a definable function $\mu(x) > 0$ such that if $|x - h(x)| < \mu(x)$ then $|d_x r(v) - d_{h(x)} r(v)| < \frac{1}{8}$.

By the above, if $\varepsilon(x) < \min\left\{\eta(x), \mu(x), \frac{1}{8(|d_x r|+1)}\right\}$ we have

$$\begin{aligned} |d_x r(u) - u| &\le |d_x r(u) - d_x r(v)| + |d_{h(x)} r(v) - u| + |d_x r(v) - d_{h(x)} r(v)| \\ &< \varepsilon(x)(|d_x r| + 1) + \frac{1}{8} < \frac{1}{4}, \end{aligned}$$

which shows that $d_x(r_{|N})$ is an isomorphism, implying that r induces a local diffeomorphism on N.

To prove that it is a diffeomorphism on N, we start by showing that the restriction $r_{|N}$ is proper. Observe that N is closed. Indeed, take $(x_i) \subset N$ such that $x_i \to x \in \overline{N}$. As $h(x_i)$ is bounded and M is closed there is a $y \in M$ such that $h(x_i) \to y$ (extracting some subsequence if necessary), which implies $x = h^{-1}(y) \in N$, showing that N is closed. Now, notice that if $|x_i| \to +\infty$ then, as $|r(x_i) - x_i| \le |h(x_i) - x_i| < \varepsilon(x_i)$, the sequence $r(x_i)$ must go to infinity as well. Since N is closed, this shows that the restriction of r to N is proper.

Hence, by Ehresmann's theorem $r_{|N}$ is a locally trivial fibration above every connected component of M. We will show that it is indeed one-to-one. For this purpose, let us fix any x in M. There is $q > 0$ such that $V_x := B(x, q) \cap M$ is simply connected and hence $r_{|N}$ is trivial above V_x.

We first show by way of contradiction that if ε is small enough (depending on x) then $r^{-1}(x) \cap N$ must be reduced to a single point. Write $r_{|N}^{-1}(x) = \{x_1, \dots, x_k\}$, suppose $k \ge 2$, and notice that

$$r_{|N}^{-1}(V_x) = V_1 \cup \dots \cup V_k$$

is a disjoint union of neighborhoods V_i of x_i in N respectively. Take now $y \in B(x, q/8) \cap M$ and set $\varepsilon_0 := \sup_K \varepsilon$, where K is some compact neighborhood of x containing all the sets that we are considering. Let us observe that

$$|h^{-1}(y) - x| \leq |h^{-1}(y) - y| + |y - x| < \varepsilon_0 + q/8,$$

which means that for $\varepsilon_0 < q/16$, we have $h^{-1}(y) \in B(x, q/4)$, and consequently

$$h^{-1}(B(x, q/8) \cap M) \subset B(x, q/4).$$

Moreover, since

$$\begin{aligned}|r(h^{-1}(y)) - x| &\leq |r(h^{-1}(y)) - h^{-1}(y)| + |h^{-1}(y) - y| + |y - x| \\ &< q/16 + q/16 + q/8\end{aligned}$$

we have $r(h^{-1}(y)) \in B(x, q/4)$, and hence

$$h^{-1}(B(x, q/8) \cap M) \subset V_1 \cup \cdots \cup V_k.$$

Since the last union is disjoint, we can assume that

$$h^{-1}(B(x, q/8) \cap M) \subset V_1.$$

Furthermore, we have

$$|h(x_2) - x| < |h(x_2) - x_2| + |x_2 - x| < 2\varepsilon_0 < q/8,$$

and hence $x_2 \in h^{-1}(B(x, q/8) \cap M)$, which contradicts the fact that V_1 and V_2 are disjoint, establishing that $r^{-1}(x) \cap N$ must be reduced to a single point.

Choose now one point in each connected component of M, say $z_1, \ldots, z_l$. By the above, for ε small enough, the set $r^{-1}(z_i) \cap N$ is reduced to a single point for all $i \leq l$. Since $r_{|N} : N \to M$ is a locally trivial fibration, it must be one-to-one.

We turn to show that $r_{|N}$ is onto. As it is a locally trivial fibration over each connected component of M, it suffices to show that $r(N)$ contains at least one point in every connected component. Take any $x \in M$ and observe that, for ε small enough, $h^{-1}(x) \in U$ and $r(h(x))$ is a point close to x, which therefore must belong to the same connected component of M as x, if ε is sufficiently small. ∎

Definition 2.2. Let M be a $\mathscr{C}^k$ submanifold of $\mathbb{R}^n$ with $k \geq 2$ (possibly infinite). Fix any $\varepsilon \in \mathcal{D}^+(M)$ and a positive integer $p \leq k$. A *definable deformation* of M is a definable family $(Z_t)_{t \in [0,1]}$ of $\mathscr{C}^p$ submanifolds $Z_t \subset \mathbb{R}^n$ with $M = Z_0$. A deformation $(Z_t)_{t \in [0,1]}$ is $(\varepsilon, \mathscr{C}^p)$ *trivial* if there exists a family of $\mathscr{C}^p$ diffeomorphisms $\varphi_t : M \to Z_t$, $t \in [0,1]$, $\mathscr{C}^p$ with respect to t and satisfying $\varphi_0(x) = x$ for all x, as well as

$$|x - \varphi_t(x)| < \varepsilon(x) \quad \text{and} \quad |u - d_x\varphi_t(u)| < \varepsilon(x), \tag{2.2}$$

for every $x \in M = Z_0$ and every unit vector $u \in T_xM$. When $(\varphi_t)_{t \in [0,1]}$ is a definable family of mappings, we say that $(Z_t)_{t \in [0,1]}$ is *definably* $(\varepsilon, \mathscr{C}^p)$ *trivial*.

Corollary 2.3. *Let $M \subset \mathbb{R}^n$ be a closed, definable, $\mathscr{C}^k$, $k \geq 2$ (possibly infinite), submanifold and let $\varepsilon \in \mathcal{D}^+(M)$. There exists $\delta \in \mathcal{D}^+(M)$ such that any $(\delta, \mathscr{C}^{k-1})$ trivial definable deformation of M is $(\varepsilon, \mathscr{C}^{k-1})$ definably trivial.*

Proof. Let $\varepsilon \in \mathcal{D}^+(M)$ and take a tubular neighborhood (U, r) of M. Take some δ sufficiently small for U to contain the closure of U_δ (see (2.1)). The derivative of r is then uniformly continuous on every bounded subset of U_δ. Let $(Z_t)_{t\in[0,1]}$ be a $(\delta, \mathscr{C}^k)$ trivial deformation, with corresponding family of diffeomorphisms $\varphi_t : M \to Z_t$. If δ is sufficiently small, by Theorem 2.1 (applied with $h := \varphi_t^{-1}$ for each t) the restriction r_t of r to each Z_t induces a definable diffeomorphism.

We are going to verify that for δ sufficiently small we have, for all $x \in Z_t$ and every unit vector $u \in T_x Z_t$ (for any t)

$$|x - r_t(x)| < \varepsilon(r_t(x)) \quad \text{and} \quad |u - d_x r_t(u)| < \varepsilon(r_t(x)). \tag{2.3}$$

Before proving these two inequalities, let us make it clear that this yields the desired fact. We may assume $\varepsilon < 1/2$. Setting $y = r_t(x) \in M$ and $v = \frac{d_x r_t(u)}{|d_x r_t(u)|} \in T_y M$ (if $x \in Z_t$ and $u \in T_x Z_t$ is a unit vector), (2.3) immediately entails $|d_x r_t(u)| \geq 1/2$ so that

$$|(r_t)^{-1}(y) - y| < \varepsilon(y) \quad \text{and} \quad |d_y (r_t)^{-1}(v) - v| < 2\varepsilon(y),$$

showing (2.2) for r_t^{-1} (up to the constant 2).

We can assume that $\delta < 1$. Observe that for such x,

$$|r_t(x) - x| = \operatorname{dist}(x, M) \leq |x - \varphi_t^{-1}(x)| \overset{(2.2)}{\leq} \delta(\varphi_t^{-1}(x)) \tag{2.4}$$

and hence

$$|r_t(x) - \varphi_t^{-1}(x)| \leq 2\delta(\varphi_t^{-1}(x)) \leq 2. \tag{2.5}$$

For $\delta'(y) := \sup\{3\delta(z) : z \in M \cap B(y, 2)\}$, $y \in M$, we deduce from (2.4) and (2.5) that

$$|r_t(x) - x| \leq \delta'(r_t(x)). \tag{2.6}$$

Now, as $x \mapsto d_x r$ is uniformly continuous on bounded sets, if δ is small enough we have on U_δ, for each unit vector $u \in T_{r_t(x)} M$,

$$|d_x r(u) - u| = |d_x r(u) - d_{r(x)} r(u)| \leq \varepsilon(r_t(x)). \tag{2.7}$$

This is almost the desired estimate (together with (2.6)). The problem is that we need such an estimate for $u \in T_x Z_t$ with $x \in Z_t \subset U_\delta$. Therefore, we are going to estimate the distance between $T_x Z_t$ and $T_{r_t(x)} M$ (see (2.8) and (2.9) below).

Denote by $\mathbb{G}_n^m$, $m = \dim M$, the Grassmannian of m-dimensional linear subspaces of $\mathbb{R}^n$, which we endow with the metric

$$\rho(P,Q) := \sup_{a\in P,\, |a|=1} \inf_{b\in Q} |a-b|.$$

Observe that

$$\rho(T_x Z_t, T_{\varphi_t^{-1}(x)} M) \overset{(2.2)}{<} \delta(\varphi_t^{-1}(x)) < \delta'(r_t(x)), \tag{2.8}$$

by definition of δ' and (2.5). Moreover, since the tangent bundle of M is at least $\mathscr{C}^1$, by (2.5), for δ sufficiently small, we have

$$\rho(T_{r_t(x)} M, T_{\varphi_t^{-1}(x)} M) < \varepsilon(r_t(x)). \tag{2.9}$$

By (2.8) and (2.9), for δ small enough we get

$$\rho(T_x Z_t, T_{r_t(x)} M) < 2\varepsilon(r_t(x)),$$

which together with (2.4) and (2.7) yields (2.3). ■

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