

VOLUME, WHITNEY CONDITIONS AND LELONG NUMBER

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ABSTRACT. This paper studies the variation of the volume of a subanalytic family of sets. More precisely we are interested in the variation of the Lelong number. We prove that the Lelong number is continuous along a stratum of a Whitney subanalytic stratification and locally Lipschitz when the stratification satisfies the Kuo-Verdier condition. This problem had been studied by G. Comte.

1. INTRODUCTION

The purpose of this paper is to prove that the Lelong number is continuous along the strata of a subanalytic Whitney stratification. The Lelong number of a real subanalytic set is given by the limit of the volume of the intersection of this set with a ball (whose radius tends to zero) divided by the volume of this ball (see (2.1)).

The idea is to give a counterpart of a well known result about complex analytic sets in the real case. A result due to Draper yields that, for a complex analytic set, the Lelong number is equal to the multiplicity. On the other hand, a famous theorem of Hironaka [Hi] states that the multiplicity is constant along the strata of a Whitney stratification of a complex analytic set. In the 80's Kurdyka and Raby [KR] extended the notion of Lelong number to the real case. Hence, the idea is also to explain that the Lelong number is a good substitute of the multiplicity for subanalytic geometry.

In [C1] G. Comte proved a result going in this direction: he proved that the Lelong number is continuous along a stratum of a subanalytic stratification which satisfies the (w) condition of Kuo-Verdier by studying the discriminant of generic projections.

We prove in section 4.1 that the Whitney condition is sufficient to ensure the continuity of the Lelong number along the strata of a subanalytic stratification. Then, we generalize our method to improve the result of [C1] by showing that Kuo-Verdier condition actually implies that the Lelong number behaves like Lipschitz function along the strata.

We show how some topological trivializations (not necessarily bi-Lipschitz), which are close to the identity, can induce some stability for the volume of subanalytic families. Such arguments may no longer be applied to sets which are not subanalytic, the finiteness of the fibers of generic projections being essential. We shall give an equimultiplicity proposition and then make use of the Cauchy-Crofton formula which relates the multiplicity to the volume. The multiplicities are compared outside a set whose measure can be bounded. For

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this purpose, we first recall the Cauchy-Crofton formula and give "some uniform bounds" for the volume of subanalytic families in section 3.

2. DEFINITIONS, NOTATIONS AND MAIN THEOREM

Notations. Throughout this paper, the letter C may stand for various strictly positive constants, when no confusion is possible. By "subanalytic set" we will mean "global subanalytic set".

For $r \in \mathbb{R}$, strictly positive, we denote by $B(0; r)$ the closed ball centered at 0 and of radius r , and by $S(0; r)$ the sphere of center 0 and radius r . Given $l \leq n$ we will write \mathbb{G}_n^l for the Grassmannian of l dimensional vector subspaces of \mathbb{R}^n .

Let \mathcal{A} be a subanalytic subset of $\mathbb{R}^n \times \mathbb{R}^m$. We will consider such a subset as a family of subanalytic subsets of \mathbb{R}^n parameterized by \mathbb{R}^m . For $U \subseteq \mathbb{R}^m$ we will denote by $\mathcal{A}|_U$ the subfamily $\{q = (x; t) \in \mathbb{R}^n \times \mathbb{R}^m / q \in \mathcal{A}, t \in U\}$, and for $t \in \mathbb{R}^m$ we will denote by \mathcal{A}_t , the fiber of \mathcal{A} at t , namely $\{x \in \mathbb{R}^n / q = (x; t) \in \mathcal{A}\}$.

We will denote by \mathcal{H}^l the l -dimensional Hausdorff measure. Given a subanalytic set A of \mathbb{R}^n and a real number ε we will denote by $A_{\leq \varepsilon}$ the neighborhood of A defined by

$$\{x \in \mathbb{R}^n / d(x; A) \leq \varepsilon\}.$$

We are going to study the Hausdorff measure of subanalytic sets. For this, if X is the germ of a subanalytic subset of \mathbb{R}^n at 0, we define the functions ψ , $\tilde{\psi}$, and $\hat{\psi}$ in the following way:

$$\psi(X; r) := \mathcal{H}^l(X \cap B(0; r)),$$

where l is the dimension of X ,

$$\tilde{\psi}(X; r) := \frac{\psi(X; r)}{\mu_l r^l}$$

where μ_l is the volume of the unit of ball in \mathbb{R}^l , and

$$\hat{\psi}(X; r) = \mathcal{H}^{l-1}(X \cap S(0; r)).$$

The limit

$$(2.1) \quad \theta(X; x) := \lim_{r \rightarrow 0} \frac{\mathcal{H}^l(X \cap B(x; r))}{\mu_l r^l}$$

is called the *density* or the *Lelong number* of X at x . The existence of this limit for a subanalytic set has been proved by Kurdyka and Raby in [KR]. Later, in [CLR] G. Comte, J. M. Lion and J. P. Rolin proved that the function ψ has a log-analytic expansion. For irreducible complex analytic sets, a result due to Draper states that this number is the multiplicity.

For the sake of clarity we recall the definitions of the Whitney (b) condition and of the Kuo-Verdier (w) condition.

Definition 2.0.1. Let A be a subanalytic subset of \mathbb{R}^n . A *subanalytic stratification* of A is a locally finite partition of A into C^2 subanalytic submanifolds of \mathbb{R}^n . We call the elements of this partition *strata*.

A couple of strata $(X; Y)$ is said to be *Whitney (b) regular at $y \in Y$* if for any sequences (x_k) and (y_k) , of points of X and Y respectively, converging to y and such that there exist

$l \in S^{n-1}$ and $\tau \in \mathbb{G}_n^p$ (where $p = \dim X$) with $\tau = \lim T_{x_k} X$ and $l = \lim \frac{x_k y_k}{|x_k y_k|}$, we have $l \in \tau$.

A couple of strata is said to be *(w) regular at* $y_0 \in Y$ if there exists a constant C such that for $x \in X$ and $y \in Y$ in a neighborhood of y_0 :

$$(2.2) \quad \delta(T_y Y; T_x X) \leq C|y - x|$$

where $\delta(E; F) = \sup_{u \in E, |u|=1} d(u; F)$ (with d the Euclidian distance) for E and F vector subspaces of \mathbb{R}^n .

It is well known that the conditions (b) and (w) both imply the topological triviality along the strata.

In this paper we prove:

Theorem 2.0.2. *Let A be a closed subanalytic set stratified by a subanalytic stratification.*

- (1) *If all the strata satisfy the Whitney (b) condition with respect to Y then the Lelong number of A is continuous along this stratum.*
- (2) *If all the strata satisfy the (w) condition of Kuo-Verdier then the Lelong number of A is locally Lipschitz along the strata.*

Remark 1. This article deals with subanalytic sets. However the results (and the proofs) can be written in the language of o-minimal geometry replacing "subanalytic" by "definable". In this case, the results assuming the Whitney (b) condition require the o-minimal structure to be polynomially bounded, whereas those assuming the Verdier condition are true over any o-minimal structure.

Given a positive subanalytic function $\alpha : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ we introduce the notion of α -approximation of the identity in the following way:

Definition 2.0.3. Let \mathcal{A} and \mathcal{B} be two families of sets of $\mathbb{R}^n \times \mathbb{R}^m$. We call α -approximation of the identity a family of mappings $h : (\mathcal{A}; 0) \rightarrow (\mathcal{B}; 0)$ of type $h(x; t) = (h_t(x); t)$ (with $h_t(0) = 0$), such that the germ of each h_t is the germ of a homeomorphism, for which we can find a strictly positive constant C such that for every t

$$(2.3) \quad |h_t(x) - x| \leq C\alpha(r; t)$$

and

$$(2.4) \quad |h_t^{-1}(x) - x| \leq \alpha(r; t)$$

for all $x \in B(0; r)$ for which the above mappings are defined, and any r sufficiently close to zero.

Let A be a subanalytic subset of \mathbb{R}^n of dimension l . For $P \in \mathbb{G}_n^l$ we will denote by π_P the orthogonal projection on P . As a consequence of Gabrielov's complement theorem [G], we know that there exists an integer N such that for any vector space P in \mathbb{G}_n^l and for all $q \in P$ either $\text{card}(\pi_P^{-1}(q) \cap A) \leq N$ or $\text{card}(\pi_P^{-1}(q) \cap A) = \infty$. This provides a finite partition of P into subanalytic subsets:

$$K_j^P(A) := \{q \in P / \text{card} \pi_P^{-1}(q) \cap A = j\},$$

$j \in \{0, \dots, N, \infty\}$. Then the l dimensional vector spaces P such that $K_\infty^P(A) = \emptyset$ are dense in \mathbb{G}_n^l .

3. PRELIMINARY RESULTS

In this section we prove some general results about the volume of subanalytic families that we will use in the next section.

A key argument of the main proofs of this paper is the Cauchy-Crofton formula. This is a classical result which relates the Hausdorff measure of a set to the measure of its projections. We recall the formula without proof. We refer the reader to [F] (Theorem 2.10.15). We do not stipulate the measure $\gamma_{l,n}$ on the Grassmannian which is involved in this formula, since we will not need anything else than the fact that it is a finite measure. The formula given in [F] actually applies to a more general class of sets than subanalytic sets. We state this formula and then rewrite it in our setting using the properties of subanalytic sets that we recalled at the end of the previous section.

Proposition 3.0.4. *Let E be a subanalytic set of dimension $l < n$. Then:*

$$(3.5) \quad \mathcal{H}^l(E) = \beta(l; n)^{-1} \int_{P \in \mathbb{G}_n^l} \int_{y \in P} N(\pi_P; E; y) d\mathcal{H}^l(y) d\gamma_{l,n}(P),$$

where $\beta(l; n)$ is a constant, and $N(\pi_P; E; y)$ is the cardinality of the fiber $\pi_P^{-1}(y)$ intersected with E .

As we said above, if E is a subanalytic set, the cardinality of finite fibers of linear projections is uniformly bounded. So, for any $P \in \mathbb{G}_n^l$ the integers j for which $K_j^P(E)$ is nonempty are bounded by an integer N and the Cauchy-Crofton formula becomes:

$$(3.6) \quad \mathcal{H}^l(E) = \beta(l; n)^{-1} \int_{P \in \mathbb{G}_n^l} \sum_{j=1}^N j \mathcal{H}^l(K_j^P(E)) d\gamma_{l,n}(P).$$

In order to study the variation of the volume of a family we need some preliminary results. They will provide uniform bounds for the volume of sets and neighborhoods of a subanalytic family which is indexed on a compact set.

We shall make use of a second basic formula of geometric measure theory. We again refer the reader to [F]. This formula, sometimes called the *coarea formula*, relates the volume of a set to the volume of the fibers of a differentiable map. We recall this formula in a particular case which will be enough for the purpose of this paper. More precisely:

Proposition 3.0.5. *Let A be a C^1 submanifold of \mathbb{R}^n of dimension $l > 1$ and let $f : A \rightarrow \mathbb{R}$ be a differentiable function. Then:*

$$(3.7) \quad \int_{y \in \mathbb{R}} \mathcal{H}^{l-1}(A \cap f^{-1}(y)) d\mathcal{H}^1(y) = \int_{x \in A} |\partial_x f| d\mathcal{H}^l(x).$$

where $|\partial_x f|$ is the norm of the gradient of f .

Note that it is enough to assume that f is smooth above almost every point in the sense that it suffices to assume that the image of the set of y for which f fails to be smooth at some point of $f^{-1}(y)$ is of measure zero.

Two preliminary propositions. The point is that the constants given by following two propositions do not depend on the parameters provided they stay in a given compact set.

Proposition 3.0.6. *Let $\mathcal{A} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be a subanalytic family of subsets of \mathbb{R}^n of dimension l and let K be a compact subset of \mathbb{R}^m . For each real number r_0 , there exists a constant C such that for any $t \in K$ and for any real number $0 \leq r \leq r_0$:*

$$\psi(\mathcal{A}_t; r) \leq Cr^l.$$

Proof. In the case $l = n$ the set $\mathcal{A}_t \cap B(0; r)$ is included in the ball $B(0; r)$. If $l < n$, it is a consequence of the case $l = n$ and (3.6). \square

The following proposition enables us to bound uniformly with respect to ε the volumes of neighborhoods of the fibers of a subanalytic family.

Proposition 3.0.7. *Let $\mathcal{A} \subseteq \mathbb{R}^l \times \mathbb{R}^m$ be a subanalytic family of subsets of \mathbb{R}^l and let K be a compact subset of \mathbb{R}^m . Assume that $\dim \mathcal{A}_t = k < l$ for any $t \in K$. Then for each real r_0 there exists a constant C such that for all t in K , for all $\varepsilon \in [0; 1]$ and for all real $0 \leq r \leq r_0$:*

$$(3.8) \quad \psi((\mathcal{A}_t)_{\leq \varepsilon}; r) \leq Cr^{l-1}\varepsilon.$$

Moreover, in the case $k = l$, then for each positive real number r_0 there exists a constant C such that for all $t \in K$ and for all $0 \leq r \leq r_0$:

$$\psi((\mathcal{A}_t)_{\leq \varepsilon}; r) \leq Cr^{l-1}\varepsilon + \psi(\mathcal{A}_t; r).$$

Proof. The norm of the gradient of the distance function to a subset of \mathbb{R}^n is equal to 1 at each point where this function is differentiable. Moreover the family

$$\mathcal{A}' = \{(x; t; \alpha) \in \mathbb{R}^l \times \mathbb{R}^{m+1} / d(x; \mathcal{A}_t) = \alpha\}$$

is a subanalytic family of subsets of \mathbb{R}^l . So, by the previous proposition, there exists a constant C such that for all $(t; \alpha) \in K \times [0; 1]$ we have: $\psi(\mathcal{A}'_{(t; \alpha)}; r) \leq Cr^{l-1}$. Thus, we can write:

$$\begin{aligned} \psi((\mathcal{A}_t)_{\leq \varepsilon}; r) &= \int_{(\mathcal{A}_t)_{\leq \varepsilon} \cap B(0; r)} d\mathcal{H}^l \\ &\leq \int_0^\varepsilon \psi(\mathcal{A}'_{(t; \alpha)}; r) d\mathcal{H}^1(\alpha) \quad (\text{by (3.7)}) \\ &\leq Cr^{l-1}\varepsilon. \end{aligned}$$

In the case where all the \mathcal{A}_t are of dimension l , we set $\mathcal{C}_t = cl(\mathcal{A}_t) \setminus Int(\mathcal{A}_t)$. Thus the sets \mathcal{C}_t form a subanalytic family of sets of dimension strictly less than l . Since $(\mathcal{A}_t)_{\leq \varepsilon} \subseteq (\mathcal{C}_t)_{\leq \varepsilon} \cup \mathcal{A}_t$, the result follows from the case $k < l$. \square

4. DENSITY OF SUBANALYTIC SETS

4.1. Density and the Whitney (b) condition. In this section we prove that the Lelong number is continuous along a stratum satisfying the Whitney (b) condition (Theorem 2.0.2). Thus, for this section, we fix a closed subanalytic set $A \subseteq \mathbb{R}^{n+m}$ of dimension l . We will assume that A is stratified by a family of C^2 subanalytic manifolds $\{Y, X_1, \dots, X_\nu\}$. Let us also assume that *all the pairs $(Y; X_i)$ are Whitney (b) regular.*

Denote by A^i the union of the strata of this stratification of dimension less than or equal to i . As in [C2], to study the behavior of the Lelong number along Y , we will assume that $Y = \{0\} \times \mathbb{R}^m$. This is no loss of generality since we may find a coordinate system Φ of Y such that $d_0\Phi = Id$. We will denote by π the orthogonal projection onto Y .

It will be convenient to work with the family:

$$\mathcal{A} = \{(x; u; t) \in A \times \mathbb{R}^m / (x; u + t) \in A\}.$$

Now the germ of \mathcal{A}_t at zero is the germ of A at $(0; t) \in Y$ (only last m variables are considered as parameters).

Similarly, we will denote by \mathcal{A}^i the family obtained from A^i :

$$\{(x; u; t) \in A^i \times \mathbb{R}^m / (x; u + t) \in A^i\}.$$

We will need another proposition to bound, uniformly in the parameter t , the variation of the Hausdorff measure of the fibers of a subanalytic family. This one relies on the assumption of the Whitney condition. More precisely:

Proposition 4.1.1. *For any compact subset V of Y there exist a strictly positive real number r_0 and a constant C such that if r and r' are positive real numbers satisfying $r' \leq r \leq r_0$ we have for any $t \in V$:*

$$|\psi(\mathcal{A}_t; r) - \psi(\mathcal{A}_t; r')| \leq Cr^{l-1}|r - r'|.$$

Proof. Let λ denote the distance to the stratum Y restricted to the regular locus of A (by regular locus we mean the points at which the set A is a smooth manifold). We wish to apply formula (3.7), so we first prove that $|\partial_q \lambda|$ tends to 1 when q tends to the origin. The norm of the gradient of the distance to Y is 1 and this vector is collinear to the secant $q\pi(q)$. Given a point $(x; t)$ of A let us denote by $X_{(x;t)}$ the stratum which contains this point. As all the strata satisfy the Whitney (b) condition and V is compact, there exists a subanalytic function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ tending to zero at the origin such that for all t in V and for all x in $X_{(x;t)}$:

$$\delta(\mathbb{R}x; T_x X_{(x;t),t}) \leq \phi(|x|)$$

where $X_{(x;t),t}$ is the fiber of $X_{(x;t)}$ at t (see Definition 2.0.1 for the definition of δ).

This implies that $|\partial_q \lambda|$ tends to 1 uniformly in t on $X|_V$ since the projection of the gradient onto the tangent space to $X_{(x;t),t}$ tends to the gradient itself when we approach the origin.

Let $\mathcal{D} = \{(x; t; r) \in S^{m+n-1}(0; r_0) \times \mathbb{R}^m \times \mathbb{R} / \frac{r}{r_0}x \in \mathcal{A}_t, r \leq r_0, t \in V\}$. If r_0 is chosen small enough the family \mathcal{D} is a subanalytic family of sets of dimension $(l-1)$ (considering t and r as parameters).

Now since the family $(\mathcal{D}_{t,r})_{\substack{t \in V \\ r \leq r_0}}$ is a subanalytic family of sets then, thanks to the Cauchy-Crofton formula, there exists a constant C such that:

$$\mathcal{H}^{l-1}(\mathcal{D}_{t,r}) \leq C.$$

But by the definition of the family $\mathcal{D}_{t,r}$ we have:

$$\mathcal{H}^{l-1}(\mathcal{D}_{t,r}) = r_0^{l-1} \frac{\widehat{\psi}(\mathcal{A}_t; r)}{r^{l-1}}.$$

This is equivalent to

$$\widehat{\psi}(\mathcal{A}_t; r) \leq Cr^{l-1}$$

for a constant C .

So, applying (3.7) on the regular locus of \mathcal{A}_t , we can write for $r' \leq r \leq r_0$ with r_0 small enough:

$$\begin{aligned} |\psi(\mathcal{A}_t; r) - \psi(\mathcal{A}_t, r')| &= \left| \int_{\mathcal{A}_t \cap B(0; r) \setminus B(0; r')} d\mathcal{H}^l \right| \\ &\leq 2 \left| \int_{\mathcal{A}_t \cap B(0; r) \setminus B(0; r')} |\partial_q \lambda| d\mathcal{H}^l(q) \right| \quad (\text{since } |\partial_q \lambda| \text{ tends to } 1) \\ &= 2 \int_{r'}^r \widehat{\psi}(\mathcal{A}_t; s) d\mathcal{H}^1(s) \\ &\leq 2Cr^{l-1} \int_{r'}^r d\mathcal{H}^1 \\ &\leq 2Cr^{l-1} |r - r'|. \end{aligned}$$

□

Remark 2. (1) In particular if A is a subanalytic set we can find real number r_0 such that for all $r' \leq r \leq r_0$:

$$|\psi(A; r) - \psi(A; r')| \leq Cr^{l-1} |r - r'|.$$

For a subanalytic family of sets we may drop the hypothesis of Whitney (b) condition if we weaken the conclusion. Namely, if we do not assume that the Whitney condition holds we may prove that: *for each t there exists r_0 such that the latter equality is fulfilled (that is to say we allow r_0 to depend on t).*

- (2) The assumption of the Whitney (b) condition can be weakened. It is clear from the proof that it is sufficient to assume that the Milnor radius is stable [BK, O]. It seems also that in this case the converse could be proved. We mean that the conclusion to Proposition 4.1.1 may imply that the Milnor radius is stable. If the Milnor radius is stable for all strata we have topological triviality. We would get a sufficient condition for topological stability.

We now prove an isotopy lemma. It is well known [Ma] that Whitney stratifications are topologically trivial along the strata. We need a precise statement in our context which will be useful in the proof of Proposition 4.1.3 to bound the measure of the set above which the respective multiplicities coincide for a generic projection.

Proposition 4.1.2. *Let $\gamma :]-\delta; \delta[\rightarrow Y$ be an analytic arc, $\gamma(0) = 0$. Then there exist a strictly positive real number η and a map $h : \mathcal{A}_{|\gamma([0; \eta])} \setminus \mathcal{B} \rightarrow (\mathcal{A}_0 \times \gamma([0; \eta])) \setminus \mathcal{B}'$ which is an α -approximation of the identity with $\alpha(r; t) = |t|^{1-e}r$ with $e < 1$, and where the families \mathcal{B}' and \mathcal{B} satisfy*

- (i) $\mathcal{B}'_t \subseteq \{q \in \mathcal{A}_0/d(q; \mathcal{A}_0^{l-1}) \leq |t|^{1-e}|q|\}$
- (ii) $\mathcal{B}_t \subseteq \{q \in \mathcal{A}_t/d(q; \mathcal{A}_t^{l-1}) \leq |t|^{1-e}|q|\}$.

for t sufficiently close to zero in the image of γ .

Proof. We construct the isotopy by integration of vector fields as in Thom-Mather isotopy lemma [Ma]. Note that the strata X_j are not necessarily Whitney (b) regular with respect to each other. So, we may only prove uniqueness of some integral curves (which are sufficiently far away from the lower dimensional strata) and we shall define the desired family of homeomorphisms on $\mathcal{A}_{|\gamma|[0;\eta]}^i$, by induction on i . So, for the induction hypothesis just replace in the proposition for a \mathcal{A} by \mathcal{A}^i and l by i .

For $i = m$ the result is clear. So, we assume that the result is true for $i \geq m$.

The stratification of A given by the X_j , $Y' := \gamma(\cdot - \delta; \delta]$ and $Y \setminus Y'$ is still (b) regular (Y' is not a smooth stratum but we will deal with $\gamma[0; \delta]$). As the arc γ constitutes a stratum of dimension one, by [K] or [OT1] the tangent spaces satisfy the following estimation:

$$\delta(T_y Y'; T_x X_j) \leq C \frac{|y - x|}{|\pi'(x)|^e}$$

(where π' is the orthogonal projection on Y') for a real number $e < 1$ in a sufficiently small neighborhood of the origin.

By standard arguments [Ve, Ma, OT1] we may obtain a stratified unit vector field v defined on A^{i+1} , tangent to Y' and tangent to the X_j 's, and satisfying:

$$(4.9) \quad |v(q) - v(\pi(q))| \leq C \frac{|q - \pi'(q)|}{|\pi'(q)|^e}$$

for some constant C in a sufficiently small neighborhood of the origin.

We may identify Y' with $\{0_{\mathbb{R}^{m-1}}\} \times \mathbb{R}$ and choose v such that $v \equiv (0; -1)$ on Y' . Denote by ϕ the one-parameter group generated by this vector field. Let $\phi = (\phi_1; \phi_2) \in \mathbb{R}^{n+m-1} \times \mathbb{R}$; we also have by (4.9) (again the reader is referred to [OT1] or [OT2]) that:

$$(4.10) \quad |\phi_1((q; u); s) - q| \leq C s^{1-e} |q|$$

for any $(q; u) \in \mathbb{R}^{n+m-1} \times \mathbb{R}$ and $s \geq 0$ for which ϕ is defined. This implies that the integral curves cannot join Y' before $s = C \frac{1}{e-1}$, and thus before $s = u$ if $0 < u < C \frac{1}{e-1}$. It is easy to see that the integral curves extend at u and are still unique.

The mapping ϕ_1 will induce the desired trivialization on A^{i+1} . We first prove that an integral curve $\phi((q; u); s)$, starting from a point $(q; u) \in \mathbb{R}^{n+m-1} \times Y'$, may not fall into A^i before $s = u$, as long as $(q; u)$ is chosen sufficiently "far away" from A^i (see (4.12)).

Note that by the induction hypothesis there exists an α -approximation of the identity h^i with $\alpha(r; t) = t^{1-e} r$ defined on $\mathcal{A}_{\{0\} \times [0; \eta]}^i$. This implies for $t \in \mathbb{R}$, considering A as a family of subsets of \mathbb{R}^{n+m-1} parameterized by the last variable:

$$(4.11) \quad d_{\mathcal{H}}(A_0^i \cap B(0; r); A_t^i \cap B(0; r)) \leq C t^{1-e} r.$$

for a constant C (where $d_{\mathcal{H}}(E; F) = \max(\sup_{x \in E} d(x; F); \sup_{y \in F} d(y; E))$ is the Hausdorff distance between sets), so that for q in A_t^{i+1} such that:

$$(4.12) \quad d(q; A_t^i) \geq 4C t^{1-e} |q|$$

we have $d(q; A_{t-s}^i) \geq 2C t^{1-e} |q|$ (applying (4.11) two times). Note that $\phi_1((q; t); s)$ is an element of A_{t-s}^{i+1} and using (4.10) we deduce

$$(4.13) \quad d(\phi_1((q; t); s); A_{t-s}^i) \geq C t^{1-e} |q|$$

when $0 \leq s \leq t$.

Therefore, by (4.10), the curve $\phi((q; t); s)$ cannot join A^i before $s = t$ if $q \in A_t^{i+1}$ satisfies (4.12). This proves the existence of the integral curves until at least $s = t$ in a neighborhood of 0 for such points $(q; t)$. So, we now set

$$\mathcal{B}_t = \{q \in \mathcal{A}_{\{0, \mathbb{R}^{m-1}\} \times [0; \eta[}^{i+1} / d(q; \mathcal{A}_{(0; t)}^i) \leq 4|q||t^{1-e}\}$$

and for $\eta > 0$ sufficiently small:

$$\begin{aligned} h : \mathcal{A}_{\{0, \mathbb{R}^{m-1}\} \times [0; \eta[}^{i+1} \setminus \mathcal{B} &\rightarrow \mathcal{A}_0 \times [0; \eta[\\ (q; 0; t) &\mapsto (\phi_1((x; u + (0; t)); t); t) \end{aligned}$$

where $q = (x; u) \in \mathbb{R}^n \times \mathbb{R}^m$.

The map h_t is a homeomorphism onto its image for each t . By (4.10) it is an α -approximation of the identity with $\alpha = t^{1-e}r$ and $e < 1$. Let $\mathcal{B}'_{(0; t)} = \mathcal{A}_0 \setminus h_t(\mathcal{A}_{(0; t)} \setminus \mathcal{B}_{(0; t)})$. By (4.13) we know that if $q \in \mathcal{A}_0$ is such that $d(q; \mathcal{A}_0^i) \geq 4t^{1-e}|q|$ then

$$d(\phi(q; -t); \mathcal{A}_0^i) \geq 3|t|^{1-e}|q|$$

and so

$$d(\phi((q; -t); \mathcal{A}_{(0; t)}^i) \geq 2t^{1-e}|q|$$

using (4.11). In consequence $\phi(q; -t) \notin \mathcal{B}_t$. Thus

$$\mathcal{B}'_{(0; t)} \subseteq \{q \in \mathcal{A}_0^{i+1} / d(q; \mathcal{A}_0^i) \leq 4t^{1-e}|q|\}$$

as claimed. \square

The next proposition is a consequence of this isotopy lemma. More precisely, we are going to see that such isotopies preserve the cardinality of generic fibers of a projection through small variations of the parameter in a subanalytic family.

Proposition 4.1.3. *For any $\varepsilon > 0$, there exists a neighborhood U_ε of 0 in Y , such that for any $P \in \mathbb{G}_{n+m}^l$ there exists a subset $K(P; \varepsilon; r; t)$ of $B(0; r) \cap P$ such that:*

$$\mathcal{H}^l(K(P; \varepsilon; r; t)) \leq \varepsilon r^l,$$

and such that for any $t \in U_\varepsilon$ and $x \in P \cap B(0; r) \setminus K(P; \varepsilon; r; t)$:

$$(4.14) \quad \text{card}(\pi_P^{-1}(x) \cap \mathcal{A}_t \cap B(0; r)) = \text{card}(\pi_P^{-1}(x) \cap \mathcal{A}_0 \cap B(0; r)).$$

Proof. We fix a strictly positive real number r . Since the family \mathcal{A} is subanalytic there exists an integer N such that for all $j \geq N$ and any P in \mathbb{G}_{n+m}^l we have $K_j^P(\mathcal{A}) = \emptyset$, and $\dim K_\infty^P(\mathcal{A}) < l$. Let

$$H(P; j; t; r) = \partial(K_j^P(\mathcal{A}_t \cap B(0; r)))$$

where ∂ denotes the topological boundary. Now set

$$H(P; t; r) = \bigcup_{j \in N \cup \{\infty\}} H(P; j; t; r)_{\leq 2\varepsilon r}.$$

The family $(H(P; t; r))_{\substack{t \in B(0; 1) \\ P \in \mathbb{G}_{n+m}^l, r \leq r_0}}$ is subanalytic and is indexed by a compact subset.

We apply Proposition 3.0.7 (identify P with \mathbb{R}^l) to get a constant C such that for $t \in$

$B(0; 1)$:

$$(4.15) \quad \psi(H(P; j; t; r)_{\leq 2\epsilon r}; r) \leq C\epsilon r^l.$$

Moreover by Proposition 4.1.1 there exists a constant C (independent of ϵ) such that

$$\mathcal{H}^l(\mathcal{A}_t \cap B(0; r) \setminus B(0; r - 2\epsilon r)) \leq C\epsilon r^l$$

for all t in $B(0; 1)$.

Let

$$\mathcal{M}_t(r) = \pi_P(\mathcal{A}_t \cap B(0; r) \setminus B(0; r - 2\epsilon r))$$

and

$$\mathcal{N}(r) = \pi_P(\mathcal{A}_0 \cap B(0; r) \setminus B(0; r - 2\epsilon r)).$$

Of course, a fortiori $\mathcal{H}^l(\mathcal{M}_t(r)) \leq C\epsilon r^l$ and $\mathcal{H}^l(\mathcal{N}(r)) \leq C\epsilon r^l$. We can derive from Proposition 3.0.7 (again identifying P with \mathbb{R}^l):

$$\mathcal{H}^l(\mathcal{M}_t(r)_{\leq 3\epsilon r}) \leq C\epsilon r^l$$

and

$$\mathcal{H}^l(\mathcal{N}(r)_{\leq 3\epsilon r}) \leq C\epsilon r^l$$

for a constant C independent of ϵ . Finally for t in \mathbb{R}^m let

$$\mathcal{Q}_t(r) = \pi_P(\mathcal{A}_t^{l-1}).$$

By Proposition 3.0.7 we again have $\psi((\mathcal{Q}_t)_{\leq \epsilon r}; r) \leq Cr^l\epsilon$ for a constant C independent of ϵ . Therefore, now we can set $K(P; \epsilon; r; t) = H(P; t; r) \cup H(P; 0; r) \cup \mathcal{M}_t(r)_{\leq 3\epsilon r} \cup \mathcal{N}(r)_{\leq 3\epsilon r} \cup (\mathcal{Q}_t)_{\leq \epsilon r} \cup (\mathcal{Q}_0)_{\leq \epsilon r}$.

By the above we have that

$$\psi(K(P; \epsilon; r; t); r) \leq Cr^l\epsilon.$$

Therefore, it suffices to check that (4.14) holds outside $K(P; \epsilon; r; t)$ for t sufficiently close to zero.

Thanks to the curve selection lemma, it suffices to check it along an analytic curve $\gamma :]-\delta; \delta[\rightarrow \mathbb{R}^m$. Let $P \in \mathbb{G}_{n+m}^l$ and $x \in P \cap B(0; r) \setminus K(P; \epsilon; r; t)$. Let $j = \text{card } \pi_P^{-1}(x) \cap \mathcal{A}_t$, with $t = \gamma(s)$, $s \geq 0$ fixed, and $j' = \text{card } \pi_P^{-1}(x) \cap \mathcal{A}_0$. We remark that by definition of $H(P; t; r)$ we have

$$d(x; \partial K_j^P(\mathcal{A}_t \cap B(0; r))) > 2\epsilon r.$$

So, over $B(x; 2\epsilon r)$, the set \mathcal{A}_t (resp. \mathcal{A}_0) is the union of j (resp. j') connected components C_1, \dots, C_j (resp. $C_1^0, \dots, C_{j'}^0$) and π_P induces an homeomorphism from C_i (resp. C_i^0) onto its image. Note that by the preceding proposition, there exists a local trivialization $h : \mathcal{A}_{|\gamma([0; \eta])} \setminus \mathcal{B} \rightarrow (\mathcal{A}_0 \times \gamma([0; \eta])) \setminus \mathcal{B}'$ which is an α -approximation of the identity with $\alpha = |t|^{1-e}r$ and $e < 1$.

But, by the definition of the α -approximations of the identity, we have:

$$|h_t^{-1}(z) - z| \leq |t|^{1-e}r,$$

for t sufficiently close to zero in the image of γ and $z \in B(0; r) \cap \mathcal{A}_t \setminus K(P; r; t; \epsilon)$ at which h_t^{-1} is defined.

Let $z \in B(x; \varepsilon r)$. Since x does not belong to $\mathcal{M}_t(r)_{\leq 3\varepsilon r}$, the ball of center x and of radius εr does not intersect the set $\mathcal{M}_t(r)_{\leq 2\varepsilon r}$. Hence, thanks to the definition of \mathcal{M} , if $z \in C_{j_0} \cap \pi_P^{-1}(z)$, with $j_0 \in \{1, \dots, j\}$, it is in $B(0; r - 2\varepsilon r)$.

Note that we have

$$\mathcal{B}_t \subseteq \{q \in \mathcal{A}_t / d(q; \mathcal{A}_t^{l-1}) \leq |t|^{1-e}|q|\}$$

which is included in $(\mathcal{Q}_t)_{\leq \varepsilon r}$ for t small enough. Thus, $h_t(z)$ is well defined. So, since $|h_t(z) - z| \leq \varepsilon r$ and $z \in B(0; r - 2\varepsilon r)$, the point $h_t(z)$ must belong to $B(0; r)$. Moreover $\pi_P(h_t(z)) \in B(x; 2\varepsilon r)$ (again since $|h_t(z) - z| \leq \varepsilon r$) and so $h_t(z)$ actually belongs to one of the C_k^0 . As C_{j_0} is connected and the C_k^0 's are disjoint, the integer k does not depend on the point z in $C_{j_0} \cap \pi_P^{-1}(B(x; \varepsilon r))$. Let $\sigma(j_0)$ be this integer.

In this way, we have defined a mapping σ from $\{1, \dots, j\}$ to $\{1, \dots, j'\}$. In order to show $j' \leq j$ it suffices to see that σ is surjective.

Let i be an integer between 1 and j' . Let z be the point of $\pi_P^{-1}(x) \cap C_i^0$ and denote $p = h_t^{-1}(z)$. Since $x \notin \mathcal{N}(r)$, the point z belongs to $B(0; r - 2\varepsilon r)$; this implies, using the inequality $|h_t^{-1}(z) - z| \leq \varepsilon r$, that the point p also belongs to $B(0; r)$.

Moreover (again using that $|h_t^{-1}(z) - z| \leq \varepsilon r$) it is clear that $\pi_P(p) \in B(x; r - \varepsilon r)$. Thus, there exists an integer i_0 such that $p \in C_{i_0}$, which implies that $\sigma(i_0) = i$.

By the symmetry of the roles of j and j' we see that the same argument can prove $j \geq j'$. \square

Now we are able to prove the first point of our main result:

Proof of (1) of Theorem 2.0.2. Let $\varepsilon > 0$. By the above proposition we know that for all $P \in \mathbb{G}_{n+m}^l$ there exists a subanalytic subset $K(P; \varepsilon; r; t)$ such that for t in a sufficiently small neighborhood of the origin in Y :

$$\psi(K(P; \varepsilon; r; t); r) \leq C\varepsilon r^l$$

and for all $x \in P \cap B(0; r) \setminus K(P; \varepsilon; r; t)$ and $t \in V$,

$$\text{card } \pi_P^{-1}(x) \cap \mathcal{A}_t \cap B(0; r) = \text{card } \pi_P^{-1}(x) \cap \mathcal{A}_0 \cap B(0; r).$$

It follows from (3.5) that for any $j \geq 0$ and for all P in \mathbb{G}_{n+m}^l . We have:

$$\psi(K_j^P(\mathcal{A}_t \cap B(0; r)) \setminus K_j^P(\mathcal{A}_0 \cap B(0; r)); r) \leq C\varepsilon r^l$$

and

$$\psi(K_j^P(\mathcal{A}_0 \cap B(0; r)) \setminus K_j^P(\mathcal{A}_t \cap B(0; r)); r) \leq C\varepsilon r^l.$$

So we get:

$$|\psi(K_j^P(\mathcal{A}_0 \cap B(0; r)); r) - \psi(K_j^P(\mathcal{A}_t \cap B(0; r)); r)| \leq C\varepsilon r^l.$$

And finally, using (3.6), we get for t sufficiently close to the origin:

$$|\psi(\mathcal{A}_t; r) - \psi(\mathcal{A}_0; r)| \leq C\varepsilon r^l.$$

This implies:

$$|\theta(A; (t; 0)) - \theta(A; 0)| \leq C\varepsilon,$$

which proves the continuity of the Lelong number at the origin. \square

4.2. The Kuo-Verdier regular case. We will apply a similar argument in the case where the strata satisfy the (w) condition of Kuo-Verdier. In this case we are going to see that the method can prove the Lipschitz character of the Lelong number along the strata. We first study the variation of the volume through an α -approximation of the identity.

So, as in the case of the Whitney (b) condition, we fix a closed subanalytic set $A \subseteq \mathbb{R}^n \times \mathbb{R}^m$. We assume that A is stratified by a family of C^2 subanalytic manifolds $\{Y, X_1, \dots, X_\nu\}$. Again we will assume that $Y = \{0\} \times \mathbb{R}^m$.

It will be convenient to work with the families:

$$\mathcal{A} = \{(x; u; t; t') \in A \times \mathbb{R}^{2m} / (x; u + t) \in A\}$$

and

$$\mathcal{B} = \{(x; u; t; t') \in A \times \mathbb{R}^{2m} / (x; u + t') \in A\}.$$

Now the germ of $\mathcal{A}_{(t;t')}$ at the origin is the germ of A at $(0; t)$ and the germ of $\mathcal{B}_{(t;t')}$ at the origin is the germ of A at $(0; t')$. Note that \mathcal{A} and \mathcal{B} are two subanalytic families of sets.

We first give a proposition of the same type of Proposition 4.1.3 in the case where we have two subanalytic families related by an α -approximation of the identity.

Proposition 4.2.1. *Let α be a subanalytic function defined on $\mathbb{R} \times \mathbb{R}^m$. Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be an α -approximation of the identity and let P be in \mathbb{G}_{n+m}^l . Then for any compact V of \mathbb{R}^m there exists a constant C and a subanalytic subset $K(P; r; t) \subseteq P$ satisfying*

$$\psi(K(P; r; t); r) \leq C\alpha(r; t)r^{l-1},$$

and such that, for any $t \in V$ we have for r sufficiently close to zero and for any $x \in P \cap B(0; r) \setminus K(P; r; t)$:

$$\text{card}(\pi_P^{-1}(x) \cap \mathcal{A}_t \cap B(0; r)) = \text{card}(\pi_P^{-1}(x) \cap \mathcal{B}_t \cap B(0; r)).$$

Proof. We define the desired set in a similar way as in Proposition 4.1.3. Nevertheless, as the situation is now different (since we now work with two families), we give details. First let:

$$H(P; j; t; r) = \partial(K_j^P(\mathcal{A}_t \cap B(0; r))) \cup \partial(K_j^P(\mathcal{B}_t \cap B(0; r))).$$

Then, as in the proof of Proposition 4.1.3:

$$(4.16) \quad \psi(H(P; j; t; r)_{\leq 2\alpha(r; t)}; r) \leq C\alpha(r; t)r^{l-1}.$$

As the family \mathcal{A} and \mathcal{B} are subanalytic subsets of $\mathbb{R}^n \times \mathbb{R}^m$ they are Whitney stratifiable. Moreover we may choose a stratification compatible with $\{0\} \times \mathbb{R}^m$. In consequence, by Proposition 4.1.1, there exists a constant C such that for r sufficiently small

$$\mathcal{H}^l(\mathcal{A}_t \cap B(0; r) \setminus B(0; r - 2\alpha(r; t))) \leq C\alpha(r; t)r^{l-1}$$

and

$$\mathcal{H}^l(\mathcal{B}_t \cap B(0; r) \setminus B(0; r - 2\alpha(r; t))) \leq C\alpha(r; t)r^{l-1}$$

for all t in V .

Let

$$\mathcal{M}_t(r) = \pi_P(\mathcal{A}_t \cap B(0; r) \setminus B(0; r - 2\alpha(r; t)))$$

and

$$\mathcal{N}_t(r) = \pi_P(\mathcal{B}_t \cap B(0; r) \setminus B(0; r - 2\alpha(r; t))).$$

Of course a fortiori we have $\psi(\mathcal{M}_t(r); r) \leq C\alpha(r; t)r^{l-1}$ and $\psi(\mathcal{N}_t(r); r) \leq C\alpha(r; t)r^{l-1}$. We can deduce using Proposition 3.0.7 that:

$$\psi(\mathcal{M}_t(r)_{\leq 3\alpha(r; t)}; r) \leq C\alpha(r; t)r^{l-1}$$

and

$$\psi(\mathcal{N}_t(r)_{\leq 3\alpha(r; t)}; r) \leq C\alpha(r; t)r^{l-1}$$

for a constant C .

So we set:

$$K(P; r; t) = H(P; \mathcal{A}_t; r) \cup H(P; \mathcal{B}_t; r) \cup \mathcal{M}_t(r)_{\leq 3\alpha(r; t)} \cup \mathcal{N}_t(r)_{\leq 3\alpha(r; t)}.$$

Now we have to show that in $P \cap B(0; r) \setminus K(P; r; t)$ the cardinal of the fibers of π_P is the same for \mathcal{A}_t and \mathcal{B}_t . This can be proved like in the proof of Proposition 4.1.3. \square

Proposition 4.2.1 enables us to compare the volumes of subanalytic families which can be related by an approximation of the identity. Indeed, the Cauchy-Crofton formula relates multiplicities to the Hausdorff measure of A . From the equality of the multiplicities in the complementary of a set whose measure is bounded explicitly, we can bound the difference between the volume of the two given sets. So we state:

Theorem 4.2.2. *Let $\alpha : \mathbb{R} \times \mathbb{R}^{2m} \rightarrow \mathbb{R}$ be a subanalytic function. Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be an α -approximation of the identity. Let V be a compact subanalytic subset of \mathbb{R}^{2m} . Then there exists a constant C such that for all t in V :*

$$|\psi(\mathcal{A}_t; r) - \psi(\mathcal{B}_t; r)| \leq C\alpha(r; t)r^{l-1}.$$

Proof. This is a consequence of the above proposition and the Cauchy-Crofton formula (3.6) (see proof of Theorem 2.0.2). \square

It is well known that Kuo-Verdier stratifications are also topologically trivial along the strata ([Ve]). We first state an isotopy theorem which gives the precise statement that we will need.

Theorem 4.2.3. *Recall that we assumed that the stratification of A satisfies the Kuo-Verdier's (w) condition. Then there exists a neighborhood U of 0 and a family of mapping $h : \mathcal{A}|_{U \times U} \rightarrow \mathcal{B}|_{U \times U}$ which preserves \mathcal{A} and which is an α -approximation of the identity with $\alpha(r; t; t') = |t - t'|r$.*

Proof. The argument is classical. We construct a vector field tangent to the strata of the stratification of A that we integrate to produce the desired family of homeomorphisms. The assumption of the (w) condition provides a flow ϕ satisfying

$$|\pi^\perp(\phi(q; s)) - q| \leq C|s|d(q; Y)$$

(where π^\perp is the projection on the the orthogonal complement to Y), for a constant C (see [Ve]). \square

We are now able to prove the Lipschitz character of the Lelong number along Kuo-Verdier regular stratifications.

Proof of (2) of Theorem 2.0.2. Let $t \in Y$ and let U be a compact neighborhood of t . By Theorem 4.2.3 there exists an α -approximation of the identity

$$h : \mathcal{A}|_{U \times U} \rightarrow \mathcal{B}|_{U \times U}$$

where $\alpha(r; t; t') = |t - t'|r$. Now Proposition 4.2.2 implies:

$$|\psi(\mathcal{A}_{(t,t')}; r) - \psi(\mathcal{B}_{(t,t')}; r)| \leq C|t - t'|r^l$$

for a constant C and t and t' sufficiently close to the origin. This implies the local Lipschitz character of the Lelong number. \square

Note that it is also possible to deduce that if $\alpha(r) \ll r$ then an α -approximation of the identity preserves the Lelong number. In consequence the Lelong number is constant along a stratum of a stratification satisfying the strict Verdier condition.

- Remark 3.** (1) Let $p : \mathbb{R}^n \rightarrow Y$ be a smooth retraction. It is also possible to prove similar results for the family of the germs of the fibers of $p|_A$.
- (2) The Lipschitz character is local since we work with subanalytic sets. Over an o-minimal structure it is possible to prove the result with a global Lipschitz constant (as long as we have a global constant in (2.2)).

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