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NOTES ON VANISHING HOMOLOGY

Dedicated to Pierre Milman, on the occasion of the second anniversary of our Working Seminar.

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Abstract. We introduce a homology theory devoted to the study of families such as semialgebraic or subanalytic families and in general of any family definable in an o-minimal structure. This also enables us to derive local metric invariants for germs of definable sets. The idea is to study the cycles which are vanishing when we approach a special fiber. We compute these groups and prove that they are finitely generated.

Résumé. On introduit une théorie d'homologie pour les familles semi-algébriques, sous-analytiques et plus généralement pour toute famille définissable dans une structure o-minimale. Cela permet aussi de définir des invariants locaux pour les singulariés définissables. L'idée est de considérer les cycles s'evanouissant lorsque l'on approche une fibre donnée. On calcule ces groupes et prouve qu'ils sont de type fini.

1. **Introduction** This note gives an outline of the work carried out in [V4], where we introduced a homology theory for families of subsets, giving information about the behavior of the metric structure of the fibers when we approach a given fiber. This enables us to construct local metric invariants for singularities. We prove that these homology groups are finitely generated when the family is definable in an o-minimal structure. This allows us, for instance, to define an Euler characteristic type invariant which is a metric invariant for germs of algebraic or analytic sets.

In [V1], the author proved a bi-Lipschitz version of Hardt's theorem [H]. This theorem pointed out that semialgebraic bi-Lipschitz equivalence is a good notion of equisingularity to classify semialgebraic subsets from the metric point of view. For this purpose, it is also very helpful to find invariants such as homological invariants.

In [GM], M. Goresky and R. MacPherson introduced intersection homology and showed that their theory satisfies Poincaré duality for pseudo-manifolds covering a quite large class of singular sets, which turned out to be of great

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interest. They also managed to compute the intersection homology groups from a triangulation, yielding that they are finitely generated. In [BB1] L. Birbrair and J.-P. Brasselet defined their admissible chains to construct the metric homology groups. Both theories select some chains by putting conditions on the support of the chains. Our approach is similar in the sense that our homology groups will depend on a *velocity* that estimates the rate of vanishing of the support of the chains.

Our method relies on the results of [V1], where the author showed existence of a triangulation enclosing the metric type of a definable singular set. To compute the vanishing homology groups we will not use the triangulation constructed in [V1] but Proposition 3.2.9 of that paper (which was actually the main step of the construction). It makes it possible for the results proved below to apply to not necessarily polynomially bounded o-minimal structures.

It is well known that, given a definable family, we may always study the evolution of the fibers by studying what is called by algebraic geometers "the generic fiber" (see Example 2.3.1 for a precise definition).

Therefore if we carry out a homology theory for definable subsets in an o-minimal structure expanding a given arbitrary real closed field, we will have a homology theory for families. This is the point of view of the present paper. Hence, even for families of subsets of $\mathbb{R}^n$, the case of an arbitrary real closed field will be required. Our approach is patterned on one of the classical homology groups as much as possible. Some statements (Theorem 2.5.1) are close to those given by Goresky and MacPherson for intersection homology but, of course, the techniques are radically different since the setting is not the same.

The admissible chains depend on a velocity which is a convex subgroup v of our real closed field R. For instance, if R is the field of real algebraic Puiseux series endowed with the order making the indeterminate T smaller than any positive real number, v may be the subgroup

$$\{x : \exists N \in \mathbb{N}, |x| \le NT^2\}. \tag{1.1}$$

The *v-admissible chains* are the chains having a "v-thin" support. Roughly speaking, if v is as above, *v-thin* subsets of R^n are the generic fibers of families of sets whose fibers collapse onto a lower dimensional subset with at least the velocity Nt^2 (if t is the parameter of the family $N \in \mathbb{N}$). For instance, let us consider the cycle given by Birbrair and Gol′dshtein's example [BG]. Namely, the subset of $X \subset R^4$ defined by

$$x_1^2 + x_2^2 = T^{2p}, \quad x_3^2 + x_4^2 = T^{2q}. \tag{1.2}$$

This set is the generic fiber of a family of tori such that the support of the generators of $H_1(X)$ collapse onto a point at rates T^p and T^q, respectively. Therefore, if for instance $p = 0$ and $q = 2$, then the 0-fiber is a circle and this family of tori is v-thin (with v as in (1.1)).

Taking all the v-admissible chains of a definable set X, we get a chain complex which immediately gives rise to the *v-vanishing homology groups* $H_j^v(X)$. We will show that these groups are finitely generated (Corollary 2.5.2).

If X is the set defined by (1.2) with v as in (1.1), the v-vanishing homology groups depend on p and q. For instance, we will prove (see Example 2.4.2) that if $p = 0$ and $q = 2$, then $H_1^v(X) = \mathbb{Q}$ (if $\mathbb{Q}$ is our coefficient group) and $H_2^v(X) = \mathbb{Q}$.

We may summarize this by saying that we get all the T^2-thin cycles of X. The group $H_j^v(X)$ is not always a subgroup of $H_j(X)$. In general we may also have cycles that do not appear in the classical homology groups, *i.e.,* those that are in the kernel of the natural map $H_j^v(X) \to H_j(X)$. The following picture illustrates an example for which such a situation occurs:

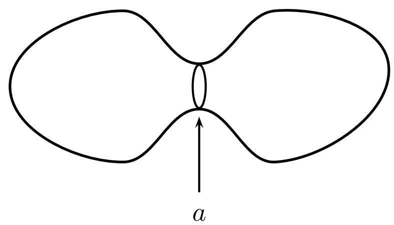

Figure 1

This picture represents the generic fiber of a family of spheres collapsing onto a point in such a way that the cycle a in the middle is collapsing much faster than the set itself. We see that we have an admissible one-dimensional chain a that bounds a two-dimensional chain which may fail to be admissible (depending on the velocity v). Therefore $H_1^v(X) \neq 0$ (while $H_1(X) = 0$).

To make this more precise an explicit example is given after the definition of the vanishing homology (Example 2.4.1).

1.1. Notations and conventions. Throughout this paper we work with a fixed o-minimal structure expanding a real closed field R. Let $\mathcal{L}_R$ be the first order language of ordered fields together with an n-ary function symbol for each function of the structure. The word definable means $\mathcal{L}_R$-definable.

The letter G will stand for an abelian group (our coefficient group). *Singular simplices* will be definable continuous maps $c\colon T_j \to X$, T_j being the j-simplex spanned by $0, e_1, \dots, e_j$ where $e_1, \dots, e_j$ is the canonical basis of R^j. Sometimes, we will work in an extension k_v of R and simplices will actually be maps $c\colon T_j(k_v) \to k_v^n$ where $T_j(k_v)$ is the extension of T_j to k_v. Given a definable set $X \subset R^n$, we denote by $C(X)$ the chain complex of definable chains with coefficients in a given group G. We will write $|c|$ for the support of a chain c.

By a *Lipschitz mapping* we will mean a mapping f satisfying

$$|f(x) - f(x')| \leq N|x - x'|$$

for some integer N. It is important to notice that we require the constant to be an integer, for R is not assumed to be archimedean. A homeomorphism $h\colon A \to R^n$ is bi-Lipschitz if h and h^{-1} are Lipschitz.

2. Construction of the vanishing homology.

2.1. The velocity v. We shall use some very basic facts of model theory. We refer the reader to [M] for basic definitions.

The vanishing homology depends on a *velocity* v that estimates the rate of vanishing of the cycles. This is a convex subgroup v of $(R;+)$, *i.e.,* $x, y \in v \implies z \in v$ for any $z \in R$ with $x \le z \le y$.

Notation. Throughout this paper, a velocity v is fixed and u is the point realizing the corresponding type in k_v.

Remark 2.1.1. Given $z \in R$ we may define a velocity $\mathbb{N}z$ by setting

$$\mathbb{N}z := \{x \in R : \exists\, N \in \mathbb{N}, |x| \le Nz\}.$$

Example 2.1.2. Let $k(0_+)$ be the field of real algebraic Puiseux series endowed with the order that makes the indeterminate T positive and smaller than any real number (see [BCR, Example 1.1.2]. Then, as in the above remark, the element T^k gives rise to a subgroup $\mathbb{N}T^k$, which is constituted by all the series z having a valuation greater or equal to k. One could also consider the velocity v defined by the set of x satisfying $|x| \le NT^k$ for any N in $\mathbb{Q}$. In the field of ln-exp definable germs of one-variable functions (in a right-hand side neighborhood of zero) one may consider the set of all the L^p integrable germs of series.

2.2. v-thin sets. We give the definition of the v-thin sets that is required to introduce the vanishing homology. We denote by $\mathbb{G}_n^j$ the Grassmaniann of j-dimensional vector spaces of R^n. Given $P \in \mathbb{G}_n^j$, we denote by π_P the orthogonal projection onto P.

Definitions 2.2.1. Let $j \le n$ be integers. A j-dimensional definable subset X of R^n is called *v-thin* if there exists $z \in v$ such that, for any $P \in \mathbb{G}_n^j$, no ball (in P) of radius z entirely lies in $\pi_P(X)$.

For simplicity we say that X is *$(j;v)$-thin* if either X is v-thin or $\dim X < j$. A set which is not v-thin will be called *v-thick*.

Note that in the above definition it is actually enough to require that the property holds for a sufficiently generic linear projection $\pi\colon R^n \to R^j$. As we said in the introduction, roughly speaking, $\mathbb{N}T^2$-thin sets of $k(0_+)^n$ are the generic fibers of one parameter families whose fibers “collapse onto a lower dimensional subset at rate at least t^2” (if t is the parameter of the family). Also, by convention $R^0 = \{0\}$ so that a 0-dimensional subset is never v-thin. This is natural in the sense that a family of points never collapses onto a lower dimensional subset.

Basic properties of $(j;v)$-thin sets.

(i) If a definable subset $A \subset X$ is $(j;v)$-thin and if $h\colon X \to Y$ is a definable Lipschitz map, then $h(A)$ is $(j;v)$-thin.

(ii) Given j, $\bigcup_{i=1}^{p} X_i$ is $(j;v)$-thin if and only if X_i is $(j;v)$-thin for any $i = 1, \dots, p$.

2.3. Definition of the vanishing homology. Given a definable set X, let $C_j^v(X)$ be the G-submodule of $C_j(X)$ generated by all the singular chains c such that $|c|$ is $(j;v)$-thin and $|\partial c|$ is $(j-1;v)$-thin as well. We endow this complex with the usual boundary operator and denote by $Z_j^v(X)$ the cycles of $C_j^v(X)$.

A chain $\sigma \in C_j^v(X)$ is said *v-admissible.* We denote by $H_j^v(X)$ the resulting homology groups which we call the *v-vanishing homology groups.*

If v is $\mathbb{N}z$ for some $z \in R$ (see Remark 2.1.1), then we will simply write $C_j^z(X)$ and $H_j^z(X)$ (rather than $C_j^{\mathbb{N}z}$ and $H_j^{\mathbb{N}z}$).

Every Lipschitz map sends a $(j;v)$-thin set onto a $(j;v)$-thin set. Thus, every Lipschitz map $f\colon X \to Y$, where X and Y are two definable subsets, induces a sequence of mappings $f_{j,v}\colon H_j^v(X) \to H_j^v(Y)$. In consequence, the vanishing homology groups are preserved by definable bi-Lipschitz homeomorphisms.

As we said in the introduction, this homology gives rise to a metric invariant for families (preserved by families of bi-Lipschitz homeomorphisms) by considering the generic fiber as described in the following example.

Example 2.3.1. With the notation of Example 2.1.2, given an algebraic family $X \subset \mathbb{R}^n \times \mathbb{R}$ defined by $f_1 = \cdots = f_p = 0$, we define the *generic fiber of* X as $X_{0_+} := \{x \in k(0_+)^n : f_1(x;T) = \cdots = f_p(x;T) = 0\}$. Hence, $H_j^v(X_{0_+})$ is a metric invariant of the family.

2.4. Two examples To illustrate the definition we provide two concrete examples.

Example 2.4.1. We first compute the homology groups on an example similar to the one sketched on Figure 1. Let

$$X(\varepsilon) := \{(x;y;z) \in k(0_+)^3 : (x - \varepsilon(1-T^4))^2 + y^2 + z^2 = 1, \varepsilon x \geq 0\}$$

for $\varepsilon = \pm 1$. Then let $X := X(1) \cup X(-1)$ and $A = X(1) \cap X(-1)$.

Let us simply consider the velocity T^2. The homology groups of X could actually be determined for any velocity. It can be computed from the definition that $H_1^{T^2}(X) \simeq \mathbb{Q}$. We also have $H_2^{T^2}(X) \simeq 0$ and $H_0^{T^2}(X) \simeq 0$.

Although this is a consequence of the definitions, the computation is not straightforward. This requires developing *ad hoc* techniques for the computation of their vanishing, such as the excision property. Details are presented in [V4].

We end by computing the vanishing homology groups of Birbrair–Goldshtein examples (compare with [BB1, §7]).

Example 2.4.2. Let X be the set defined by (1.2) assume that $p < q$. Let us compute, for instance, the vanishing homology groups for the velocity T^q. This example is easier, for we may derive the vanishing homology groups from the classical homology groups of X, since it is $\mathbb{N}T^q$-thin. This implies that the inclusion $H_2^{T^q}(X) \to H_2(X)$ is an isomorphism and that the inclusion $H_1^{T^q}(X) \to H_1(X)$ is one-to-one. Therefore $H_2^{T^q}(X) \simeq \mathbb{Q}$ and $\dim H_1^{T^q}(X) \leq 2$. Actually, one generator of $H_1(X)$ has a representative with T^q-thin support and every 1-chain representing a different class has a support whose length is clearly bigger than T^p. This proves that $\dim H_1^{T^q}(X) = 1$.

2.5. The main result. We now come to the main theorem; we compute the vanishing homology groups. We express them in terms of the homology groups of some sets X_i's (called "basic sets"). This implies in particular that they are finitely generated. In particular, this enables us to define an Euler characteristic type invariant χ_v (defined as usual) which is a definable metric invariant.

Theorem 2.5.1. *For any $X \subset R^n$ closed definable, there exist some definable subsets of X, $X_0 \subset \cdots \subset X_{d+1} = X$ such that*

$$H_j^v(X) \simeq \mathrm{Im}(H_j(X_j) \longrightarrow H_j(X_{j+1})),$$

where the arrow is induced by inclusion and Im *stands for image.*

Corollary 2.5.2. *For any closed definable subset X, the vanishing homology groups $H_j^v(X)$ are finitely generated.*

Proof. Outline of the proof of Theorem 2.5.1. The proof is constructive in the sense that we explicitly exhibit the basic sets X_i's. Part of the difficulty is due to the fact that all through the proof we will have to discuss whether or not distances belong to v. As v is not definable, belonging to v will not give rise to definable subsets. To overcome this difficulty we shall add a point u at the end of v. We extend the language $\mathcal{L}_R$ to the language $\mathcal{L}_R(u)$ by adding an extra symbol u. We then define a 1-type by saying that a sentence $\psi(u) \in \mathcal{L}_R(u)$ is in this type if and only if the set $\{x \in R : \psi(x)\}$ contains an interval $[a\,;b]$ with $a \in v$ and $b \notin v$. This type is complete due to the o-minimality of the theory. We will denote by k_v an $\mathcal{L}_R$-elementary extension of R realizing this type, and by X_v the extension of X to k_v. We may "extend" the group v in a natural way

$$w := \{x \in k_v : \exists\, y \in v, |x| \leq y\}.$$

Thus, the first step of the proof is to show that the v-vanishing homology of X coincides with the w-vanishing homology of the extension of X to k_v. We also prove that the $\mathbb{N}u$-vanishing homology groups of X_v are isomorphic to the v-vanishing homology groups of X. Namely, we show the following.

Lemma 2.5.3. *For any j, $H_j^w(X_v) \simeq H_j^u(X_v) \simeq H_j^v(X)$.*

This means that we may work with k_v. Being u-thin is somewhat weaker than being w-thin but the above lemma ensures that we will have an isomorphism between the vanishing homology groups if we eventually get a $\mathbb{N}u$-admissible chain.

In the case of intersection homology the basic sets may be defined as the allowable simplices of a barycentric subdivision of a triangulation of the considered set X. Unfortunately, this will be more complicated for us: it is not enough to define the X_i's as the cells of dimension i which are v-admissible. We need to construct a very specific cell decomposition. Therefore, the first step is to prove that, given some subsets $Y_1, \dots, Y_m$ of R^n, there exists a cell decomposition of R^n compatible with the Y_i's and such that every n-dimensional cell is delimited by graphs of Lipschitz functions $\xi_1 < \xi_2$ in such a way that $(\xi_2 - \xi_1 - u)$ has a constant sign on the cell. The construction relies on techniques developed in [V1].

We define the sets X_i's by induction on i (initializing by $X_{-1} = \varnothing$). Fix a cell decomposition $\mathcal{C}$ compatible with the $X_{j,v}$'s $j < i$ such that every cell is delimited by Lipschitz functions whose difference is comparable with u (for the order relation $\leq$) on the cell. Then, roughly speaking, the set X_i may be defined as the union of the cells which are $(i;v)$-thin.

As often with homological invariants, we construct the desired isomorphisms by means of homotopy operators. The vanishing homology is not a homotopy invariant. It is preserved by Lipschitz homotopies but these are very hard to get. We will construct a homotopy carrying a given w-admissible simplex σ onto an $\mathbb{N}u$ admissible simplex. Thanks to the lemma above this is enough for our purpose.

It is actually too difficult to construct a homotopy for any admissible simplex σ. We are able to construct this homotopy only for *strongly admissible simplices*. These are simplices $\sigma\colon T_j \to X_v$ satisfying for some q and for any λ such that $(x + \lambda e_q) \in T_j$:

$$(\sigma(x) - \sigma(x + \lambda e_q)) \in v. \tag{2.1}$$

Strongly v-admissible implies v-admissible. The difference is that the condition is required on the mapping instead of its image. We first prove that every v-vanishing homology class has a strongly admissible representative. This is achieved by introducing and constructing v-admissible rectilinearizations (see [V4, Definition 2.3.1; Theorem 2.3.3]) which may be considered as a weak notion of triangulation with an extra property close to (2.1). The proof of existence of rectilinearizations of v-thin sets follows an idea which is similar to the one used to yield existence of Lipschitz triangulations in [V1].

We finally construct a homotopy carrying a given strongly w-admissible chain onto a $\mathbb{N}u$-admissible one of the cell decomposition $\mathcal{C}$. This is performed by sending the vertices of T_j onto some zero dimensional cells of $\mathcal{C}$. Doing this sufficiently carefully, (2.1) ensures that the simplex will remain $\mathbb{N}u$-admissible if the cell decomposition is as above, *i.e.,* if every cell is either u-thick of u-thin.

The above isomorphisms of the lemma make it possible to finish the proof of Theorem 2.5.1. □

2.6. Local invariants for singularities. In [V2], we introduced the link for a semialgebraic metric space. Let us recall its definition. We recall that we denote by $k(0_+)$ the field of algebraic Puiseux series endowed with the order that makes the indeterminate T positive and smaller than any real number. Given the germ at 0 of a semialgebraic set X, let $L_X := \{x \in X_{k(0_+)} : |x| = T\}$, where $T \in k(0_+)$ is the indeterminate and $X_{k(0_+)}$ the extension of X to $k(0_+)$.

THEOREM 2.6.1. *For any convex subgroup $v \subset k(0_+)$, the groups $H_j^v(L_X)$ are semialgebraic bi-Lipschitz invariants of X.*

Note that by Corollary 2.5.2 these groups are finitely generated and that $\chi_v(L_X)$ is a semialgebraic bi-Lipschitz invariant of the germ X.

REMARK 2.6.2. We assumed in this section that X is a semialgebraic set because this was the setting of [V2]. Nevertheless, the main ingredient of the proof of the above theorem is Theorem 5.1.3 of [V1]. As this theorem holds over any polynomially bounded o-minimal structure, the above corollary is still true in this setting as well. The metric type of the link L_X may fail to be a metric invariant of the singularity when the set is definable in a non-polynomially bounded o-minimal structure as it is shown by the following example.

EXAMPLE 2.6.3. Let

$$X := \{(x\,;y) \in \mathbb{R}^2 : |y| = e^{-1/x^2}\} \quad \text{and} \quad Y = \{(x\,;y) \in \mathbb{R}^2 : |y| = e^{-2/x^2}\}.$$

Note that X and Y are both definable in the $\ln - \exp$ structure (see [vDS], [LR], [W]). Furthermore X and Y are definably bi-Lipschitz homeomorphic. However the links of X and Y are constituted by two points of $k_{0_+}^2$ (where k_{0_+} is the corresponding residue field) whose respective distances are clearly not equivalent.

Note that a revolution of these subsets about the x-axis provides two subsets whose links have different vanishing homology groups (for a suitable velocity).

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